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Some remarks on the Gelfand–Cetlin system

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Abstract
In the first section of this paper, we show that the functions in involution of the Gelfand–Cetlin system can be obtained from a λ-parametric Lax equation. In the second section, we observe that the Gelfand–Cetlin system has no obstructions to global action–angle coordinates, and we give an explicit expression of global (action) angle coordinates. In the third section, we remark the fact that the Gelfand–Cetlin system is obtained via a nesting of superintegrable systems, and show they all present a non-vanishing Chern class.

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Introduction

The Gelfand–Cetlin system is a completely integrable system well known in the community of symplectic geometers. The symplectic manifold $\mathcal{O}_n$ hosting this system is a regular $U(n)$-coadjoint orbit, and the functions in involution, functionally independent in $\mathcal{O}'_n$, an open subset of $\mathcal{O}_n$, are the eigenvalues of a family of nested minors.

A completely integrable system gives rise to a possibly singular torus fibration. Once the subset of singular fibres is excised, the torus bundle gives rise to cohomology classes—the obstructions to global action–angle coordinates. In the case of the Gelfand–Cetlin system, all obstructions to global action–angle coordinates vanish; we will provide, in section 2, the expression of global angles (the global actions being the eigenvalues).

In the third section, we will consider the superintegrable systems used to obtain the Gelfand–Cetlin completely integrable system. This superintegrable system is defined in $\mathcal{O}''_n$, an open dense subset of $\mathcal{O}_n$ strictly larger than $\mathcal{O}'_n$. The superintegrable Gelfand–Cetlin system presents an obstruction to global angles that is represented by a non-vanishing Chern class.

Given a dynamical system on a Lie algebra one can try, following Manakov [14], to produce integrals of motion of the system by writing the equations of motion as a λ-parametric

Lax equation, and using the conservation laws associated with all ODEs written in commutator form. In the first section, we produce a Lax equation for a suitable dynamical system on a regular $U(n)$-coadjoint orbit, and we show that this equation produces precisely the functions in involution of the Gelfand–Cetlin system.

1. Gelfand–Cetlin system and Lax equations

1.1. Complete integrability

In the classical definition of completely integrable system, one assumes as given a $2d$-dimensional symplectic manifold with d independent Poisson-commuting functions. The class of completely integrable systems has been extended by Nekhoroshev [15] to the non-commutative case—more than d functions but non-trivial Poisson-commutation relations—and has been finally presented by Fomenko and Mischenko [9] in the form we will be using. A *completely integrable system* is a Poisson submersion

$$(K^n \to) M^{2d} \to P^{2d-n}$$

of a symplectic manifold M onto a regular Poisson manifold P of rank $2d - 2n$, with compact and connected fibres K. We recall that most authors call a completely integrable, a system for which $d = n$ (P has a trivial Poisson structure), and call *superintegrable* or *non-commutatively integrable* a system for which $d > n$ (P has a non-trivial Poisson structure). We also recall that a regular Poisson manifold is a Poisson manifold whose bivector field has constant rank.

To distinguish between this purely geometrical set-up of complete integrability and the environment in which this theory acquires its full significance, we will call a *completely integrable dynamical system* a completely integrable system with a given Hamiltonian function f on M that Poisson-commutes with the pull-back of any function on P. What makes completely integrable dynamical systems so interesting is that they often appear in physical problems (see [7]), and their evolution can be easily described.

Theorem (action–angle coordinates [4, 9])**.** *Given a completely integrable dynamical system, every point of M has an open neighbourhood U, saturated with respect to the projection on P, and coordinates $(p, q, a, \varphi) : U \to \mathbb{R}^{d-n}_p \times \mathbb{R}^{d-n}_q \times \mathbb{R}^n_a \times T^n_\varphi$ such that the symplectic form is*

$$\sigma = \mathrm{d}p_r \wedge \mathrm{d}q^r + \mathrm{d}a_i \wedge \mathrm{d}\varphi^i$$

and the evolution over time of the dynamical system defined by the symplectic gradient of f is expressed by the equations

$$\dot{a} = 0 \qquad \dot{p} = 0 \qquad \dot{q} = 0 \qquad \dot{\varphi} = \omega_f(a).$$

It is well established to call *action functions* the functions a, *angle functions* the circle-valued functions φ, and *frequencies' map* the function ω_f.

1.2. The Gelfand–Cetlin system

Let $U(n)$ be the group of unitary matrices; its Lie algebra $\mathfrak{u}(n)$ is the vector space of skew-Hermitian matrices, while its dual Lie algebra $\mathfrak{u}^*(n)$ can be canonically identified with the space of Hermitian matrices $\mathrm{i}\mathfrak{u}(n)$, the natural duality between $\mathfrak{u}(n)$ and $\mathrm{i}\mathfrak{u}(n)$ being given by the formula

$$\langle H, A\rangle = \mathrm{i}\,\mathrm{Tr}\, HA.$$

The generic regular coadjoint orbit in $\mathrm{i}\mathfrak{u}(n)$ is an orbit through the matrix diag $(\mu_1, \ldots, \mu_n)$, for some real numbers $\mu_1 < \cdots < \mu_n$; we will denote such a coadjoint orbit by $\mathcal{O}_n$. The manifold $\mathcal{O}_n$ has dimension $n(n-1)$, and the vectors tangent to $\mathcal{O}_n$ at a point H are of the form $\mathrm{ad}_X H = [X, H]$, with X an element of $\mathfrak{u}(n)$ (a skew-Hermitian matrix).

Every dual Lie algebra $\mathfrak{g}^*$ has a canonical Poisson structure, called the Lie–Poisson structure, whose symplectic leaves are the G-coadjoint orbits. The Lie–Poisson symplectic structure of the manifold $\mathcal{O}_n$ can be expressed with the very simple formula

$$\sigma_H\big(\mathrm{ad}_{X_1} H, \mathrm{ad}_{X_2} H\big) = \mathrm{i}\,\mathrm{Tr}(H[X_1, X_2]).$$

What is commonly called the Gelfand–Cetlin system was introduced by Thimm [18] (see also [11]) as a (commutative) completely integrable system defined on $\mathcal{O}_n$—a regular coadjoint orbit of the group $U(n)$. In this system, the functions in involution are the coefficients of the characteristic polynomials of a family of nested principal minors. To be more precise, let H be an element of $\mathcal{O}_n$. By $H^{(i)}$ we denote the principal minor of H obtained by cancelling the last $n-i$ rows and last $n-i$ columns. By τ_j^i we denote the coefficient of the $(i-j)$th power of the variable in the characteristic polynomial of the matrix $H^{(i)}$. The functions $\tau_j^n(H)$ are constant along the coadjoint orbit, and will be denoted simply by $\tau_j(H)$. By μ_p^i we denote the ordered $\big(\mu_p^i \leqslant \mu_{p+1}^i\big)$ eigenvalues of $H^{(i)}$, with v_p^i their respective eigenvectors. To compactify notations, we shall write $\vec{\tau}$ for $\big(\tau_1^1, \tau_1^2, \tau_2^2, \ldots, \tau_1^{n-1}, \ldots, \tau_{n-1}^{n-1}\big)$ and $\vec{\mu}$ for $\big(\mu_1^1, \mu_1^2, \mu_2^2, \ldots, \mu_1^{n-1}, \ldots, \mu_{n-1}^{n-1}\big)$.

The regular points of $\vec{\tau}$ are precisely the set in which $\vec{\mu}$ is smooth. We will denote by $\mathcal{O}_n'$ the subset of $\mathcal{O}_n$ in which the function $\vec{\mu}$ is smooth; so that the Gelfand–Cetlin completely integrable system is the fibration $\mathcal{O}_n' \to \mathbb{R}^{n(n-1)/2}$ with $\vec{\mu}$ as projection map.

The minimax theorem can be used to obtain conditions on the possible values of the functions μ_j^i; such conditions are referred to as the *Gelfand–Cetlin pattern*. The statements in this proposition are either proved in [11] or are trivial.

Proposition 1.1. *Let H be a Hermitian matrix and μ_p^i the ordered eigenvalues of $H^{(i)}$, then note the following:*

- *The eigenvalues must satisfy the inequalities*
$$\mu_p^{i+1} \leqslant \mu_p^i \leqslant \mu_{p+1}^{i+1}.$$
- *The above inequalities define the image of the smooth map $\vec{\mu}$. In other words, the image $\vec{\mu}(\mathcal{O}_n')$ is the set $\big\{\big(\mu_p^i\big)\big|\mu_p^{i+1} < \mu_p^i < \mu_{p+1}^{i+1}\big\}$. Therefore, the set $\vec{\mu}(\mathcal{O}_n')$ is diffeomorphic to the set $(0,1)^{n(n-1)/2}$ (by a polynomial map).*
- *The smooth functions μ_p^i are functionally independent precisely on $\mathcal{O}_n' \subset \mathcal{O}_n$, where $\mathcal{O}_n'$ is the open dense subset of $\mathcal{O}_n$ defined by the inequalities*
$$\mu_p^{i+1} < \mu_p^i < \mu_{p+1}^{i+1}.$$

One last requirement the submersion $\vec{\mu} : \mathcal{O}_n \to \mathbb{R}^{n(n-1)/2}$ has to satisfy to define a completely integrable system is that its pre-images must be compact and connected. Compactness is obvious, connectedness can be proved with a straightforward argument.

Lemma 1.2. *The pre-images of $\vec{\mu}$ (and so those of $\vec{\tau}$) are connected.*

Proof. The proof of this fact can be obtained by induction on n. If $n = 1$ then the statement is trivially true. Let A and B be matrices of order n, with the same eigenvalues and such that $\vec{\tau}(A) = \vec{\tau}(B)$. By inductive hypothesis, one knows that there exists a path $U(t)$ in $U(n-1)$

such that $U(0) = 1$, $U(1)^* B^{(n-1)} U(1) = A^{(n-1)}$ and $\vec{\tau}(U(t)^* B^{(n-1)} U(t)) \equiv \vec{\tau}(A^{(n-1)})$. We are left to show that there exists a path connecting the matrices

$$\begin{pmatrix} A^{(n-1)} & a \\ a^* & a_{n,n} \end{pmatrix} \qquad \begin{pmatrix} A^{(n-1)} & b \\ b^* & b_{n,n} \end{pmatrix}.$$

The condition that the eigenvalues of the two matrices are the same, imposes that $a_{n,n} = b_{n,n}$ and that $b = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\vartheta_1}, \ldots, \mathrm{e}^{\mathrm{i}\vartheta_{n-1}})a$. The path of matrices that allows us to conclude is

$$t \mapsto \begin{pmatrix} A^{(n-1)} & \mathrm{diag}(\mathrm{e}^{\mathrm{i}t\vartheta_1}, \ldots, \mathrm{e}^{\mathrm{i}t\vartheta_{n-1}})a \\ \mathrm{diag}(\mathrm{e}^{\mathrm{i}t\vartheta_1}, \ldots, \mathrm{e}^{\mathrm{i}t\vartheta_{n-1}})a^* & a_{n,n} \end{pmatrix}. \qquad \square$$

1.3. The equation in commutator form

Beginning in the mid-1960s, particular attention has been given to the first-order differential equation which can be written in the form $\dot{M} = [\mathcal{L}(M), M]$, with M an element in some algebra of matrices $\mathfrak{g}$ and with $\mathcal{L}$ a function from $\mathfrak{g}$ to $\mathfrak{g}$. Equations of this kind are called *Euler–Arnol'd equations*, or *Euler–Poincaré equations* or *equations in commutator form*.

It is a remarkable observation of Arnol'd [2] that, when a Lie algebra $\mathfrak{g}$ admits an Ad-invariant scalar product—and hence a G-equivariant isomorphism between the Lie algebra and its dual—any Hamilton equation defined on the dual Lie algebra endowed with its Lie–Poisson structure can be rewritten on the Lie algebra in commutator form. For the sake of completeness, we include the short proof of this fact.

Lemma 1.3([2]). *Assume that a Lie algebra $\mathfrak{g}$ has a non-degenerate Ad-invariant scalar product $(-,-)$. Then, the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ allows one to write the Lie–Poisson structure in $\mathfrak{g}^*$ as a Poisson structure in $\mathfrak{g}$. The symplectic leaves of such a structure are the adjoint orbits, the space tangent to the symplectic leaves at a point M is spanned by the vectors $\mathrm{ad}_X M = [X, M]$ and, if M, X, Y are elements of $\mathfrak{g}$, the symplectic form of the leaf through M is the bilinear map*

$$\sigma_M([X, M], [Y, M]) = (M, [X, Y]).$$

Let f be a function defined on $\mathfrak{g}$, then the Hamilton equation

$$\dot{M} = \underline{f}_{\mathfrak{g}}(M)$$

can be rewritten in commutator form as

$$\dot{M} = [\nabla f(M), M]$$

where $\nabla f(M)$ is the gradient vector field defined by $(\nabla f(M), N) = \langle \mathrm{d}f(M), N\rangle$ and $\underline{f}_{\mathfrak{g}}$ is the Hamiltonian vector field obtained by contracting the Lie–Poisson structure of $\mathfrak{g}$ with the 1-form $\mathrm{d}f$.

Proof. All we have to show is that, for any X in $\mathfrak{g}$,

$$\sigma_M(\underline{f}_{\mathfrak{g}}(M), [X, M]) = \sigma_M([\nabla f(M), M], [X, M]).$$

The left-hand side, using the definition of σ, is $\langle \mathrm{d}f(M), [X, M]\rangle$; by definition of gradient this is equal to $(\nabla f(M), [X, M])$. The right-hand side, using the definition of σ, is $(M, [\nabla f(M), X])$, which is, by Ad-invariance, $([X, M], \nabla f(M))$. $\square$

Once a system is written in commutator form, one can compute some integrals of motion. In fact, the vector $[\mathcal{L}(M), M]$ is tangent to the adjoint orbit at M. Hence, the eigenvalues of the time-dependent matrix M are independent of t. One can push this idea one step further, and try to obtain other integrals of motion representing[1], if possible, an Euler–Poincaré–Arnol'd equation in the form of a λ-*parametric Lax equation*, also called λ-*parametric deformation* of the Euler–Poincaré–Arnol'd equation.

A Lax equation is a first-order differential equation $\dot{M}_\lambda = [M_\lambda, N_\lambda]$ with M_λ and N_λ polynomials in λ with coefficients in a Lie algebra. The integrals of motion can be obtained using the fact that the characteristic polynomial of the time-dependent matrix M_λ is an invariant of motion. This means that the characteristic polynomial $(\det M_\lambda - \mu\mathbb{I})$ is a polynomial in λ, μ whose coefficients are constants of motion.

Lax equations have been used to produce integrals of motion for various dynamical systems: the Euler–Poinsot top [14], the Toda lattice [1], the Lagrange and symmetric top [16, 17], and some others. The goal of the next two subsections is to define a Hamiltonian in $\mathcal{O}_n$ whose Hamilton equation can be written in Lax form, and to obtain the action functions of the Gelfand–Cetlin system as integrals of motion of that Hamiltonian system.

1.4. A Hamiltonian for the Gelfand–Cetlin system

We will now determine a Hamiltonian function whose Hamiltonian flow is $T^{n(n-1)/2}$-dense. It is best to seek the Hamiltonian among functions that are low-degree polynomials in the coefficients of H. The $U(n-1)$-equivariant linear Hamiltonians have periodic flows; the next simplest Hamiltonians are quadratic polynomials in the coefficients of H.

Proposition 1.4. *Let $\mathcal{O}$ be a regular $U(n)$-coadjoint orbit and let*

$$f : \mathcal{O} \to \mathbb{R} \qquad H \mapsto \mathrm{Tr}(H^{(n-1)})^2 + \mathrm{Tr}(H^{(n-2)})^2 + \cdots + \mathrm{Tr}(H^{(1)})^2. \tag{1}$$

Then f commutes with all the functions of the Gelfand–Cetlin completely integrable system and its Hamiltonian flow is a $T^{n(n-1)/2}$-dense dynamical system.

Proof. Once observed that the Hamiltonian f is nothing other than the function $f(H) = \sum \left(\mu_p^i(H)\right)^2$, one can use the theorem on action–angle coordinates which states that, given a Hamiltonian function f commuting with the functions of a completely integrable system, and chosing a family of local action–angle coordinates (a, φ) for this system, the Hamilton equation associated with f is

$$\dot{a} = 0 \qquad \dot{\varphi} = \omega_f(a).$$

In our case, the period function is

$$\omega_f\left(\mu_{n-1}^{n-1}, \ldots, \mu_1^1\right) = \left(2\mu_{n-1}^{n-1}, \ldots, 2\mu_1^1\right)$$

and its Jacobian is two times the identity matrix, which has maximal rank at every point of $\mathcal{O}'_n$. □

1.5. The Lax equation

We have shown that f, the quadratic Hamiltonian in (1), defines a flow which is generically dense in the Lagrangian torus-foliation called 'the Gelfand–Cetlin system'. Applying

[1] The word *representing* is purposely vague, since the technique to obtain a Lax equation from an Euler–Poincaré–Arnol'd equation needs to be invented case by case.

lemma 1.3 to this function, one obtains that $\nabla f(H) = \mathcal{L}^{(n-1)}(H) + \cdots + \mathcal{L}^{(1)}(H)$, where $\mathcal{L}^{(j)}$ is the linear map from $\mathfrak{u}(n)$ into itself such that

$$\mathcal{L}^{(j)}(H) = \begin{pmatrix} H^{(j)} & 0 \\ 0 & 0 \end{pmatrix}.$$

The Hamilton equation associated to f can be written in commutator form as

$$\dot{H} = [\mathcal{L}^{(n-1)}(H) + \cdots + \mathcal{L}^{(1)}(H), H]. \tag{EPA}$$

The dynamical system (EPA) admits a λ-parametric isospectral deformation, which is

$$(H + \lambda\mathcal{L}^{(n-1)}(\mathbb{I}_n))^\bullet = [\mathcal{L}^{(n-1)}(H) + \cdots + \mathcal{L}^{(1)}(H), H + \lambda\mathcal{L}^{(n-1)}(\mathbb{I}_n)]. \tag{Lax}$$

It is in fact obvious that $[\mathcal{L}^{(j)}(H), \lambda\mathcal{L}^{(n-1)}(\mathbb{I}_n)] = 0$ for every j, and $(H+\lambda\mathcal{L}^{(n-1)}(\mathbb{I}_n))^\bullet = \dot{H}$. These equations imply the identity of the two ordinary differential equations (EPA) and (Lax).

It follows that the characteristic polynomial of the λ-dependent matrix $H + \lambda\mathcal{L}^{(n-1)}(\mathbb{I}_n)$ is an invariant of motion. Hence, its eigenvalues are invariants of motion. In particular, the product of the eigenvalues, which is a polynomial of degree n in λ, has $n - 1$ non-trivial coefficients of the powers of λ that are integrals of motion.

Lemma 1.5. *From equation (Lax) it follows that the functions $\tau_i^{n-1}(H)$, $i = 1, \ldots, n-1$ are integrals of motion of the Hamiltonian system*

$$\dot{H} = \underline{f}_{\mathfrak{g}}(H)$$

where f is the function in proposition 1.4.

Proof. The coefficients of the characteristic polynomial of the matrix $H + \lambda\mathcal{L}^{(n-1)}(\mathbb{I}_n)$ are polynomials in λ and are integrals of motion for the dynamical system. It follows that the coefficients of the powers of λ in these polynomials are integrals of motion for the Hamiltonian system.

Using notation previously defined,

$$\tau_n(H + \lambda\mathcal{L}^{(n-1)}(\mathbb{I}_n)) = \tau_n(H) + \lambda\left(\tau_{n-1}(H) - \tau_{n-1}^{n-1}(H)\right) + \cdots + \lambda^i\left(\tau_{n-i}(H) - \tau_{n-i}^{n-1}(H)\right) + \cdots + \lambda^{n-1}\left(\tau_1(H) - \tau_1^{n-1}(H)\right).$$

This term alone gives the integrals listed in the lemma. The other coefficients of the characteristic polynomial are

$$\tau_{n-j}(H + \lambda\mathcal{L}^{(n-1)}(\mathbb{I}_n)) = \sum_{1\leqslant i_1<\cdots<i_{j-1}<n} \tau_{n-j}(H^{\widehat{I,n}} + \lambda\mathbb{I}_{n-j}) + \sum_{1\leqslant i_1<\cdots<i_j<n} \tau_{n-j}(H^{\hat{I}} + \lambda\mathcal{L}^{(n-j-1)}(\mathbb{I}_{n-j})) = \cdots$$

(the letter I represents the list of numbers $i_1, \ldots, i_j$, by $H^{\hat{I}}$ we denote the minor obtained from H by cancelling rows and columns corresponding to the indices in I)

$$\cdots = \tau_{n-j}(H) + \lambda\left(\binom{j+1}{1}\tau_{n-j-1}(H) - \tau_{n-j-1}^{n-1}(H)\right) + \cdots + \lambda^l\left(\binom{j+l}{l}\tau_{n-j-l}(H) - \tau_{n-j-l}^{n-1}(H)\right) + \cdots + \lambda^{n-j-1}\left(\binom{n-1}{n-j-1}\tau_1(H) - \tau_1^{n-1}(H)\right) + \lambda^{n-j}.$$

These polynomials do not give any new integral of motion, since their coefficients involve combinations of $\tau_j(H)$ and $\tau_j^{n-1}(H)$. □

This result cannot be considered satisfactory, since we obtained $n-1$ integrals of motion for a system with $n(n-1)/2$ degrees of freedom. We will hence rewrite the Hamiltonian system in commutator form on a different Lie algebra, write a λ-parametric perturbation of it, obtain $n(n-1)/2$ integrals of motion, and observe that such integrals are the involutory functions of the Gelfand–Cetlin system.

Lemma 1.6. *The Lie algebra* $\mathfrak{u}(n)$ *can be identified (as a vector space) with the subset* $\mathcal{S}$ *of* $\oplus_{j=1}^{n}\mathfrak{u}(j) \subset \mathfrak{u}(n(n-1)/2)$ *consisting of all the matrices of the form*

$$\mathcal{H} = \left(\begin{array}{c|c|c|c} H & 0 & 0 & 0 \\ \hline 0 & H^{(n-1)} & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & H^{(1)} \end{array}\right)$$

with H *a skew-Hermitian* $n \times n$ *matrix.*

This statement does not require any proof. What needs some explanations is the following:

Lemma 1.7. *The Euler–Poincaré–Arnol'd equation in* $\mathcal{O}_n$ *associated with the Hamiltonian* f *can be rewritten on the vector space* $\mathcal{S}$ *as*

$$\dot{\mathcal{H}} = \left[\left(\begin{array}{c|c|c} \begin{array}{c}\mathcal{L}^{(n-1)}(H)+ \\ \vdots \\ +\mathcal{L}^{(1)}(H)\end{array} & 0 & 0 \\ \hline 0 & \begin{array}{c}\mathcal{L}^{(n-2)}(H^{(n-1)})+ \\ \vdots \\ +\mathcal{L}^{(1)}(H^{(n-1)})\end{array} & 0 \\ \hline 0 & 0 & \ddots \end{array}\right), \mathcal{H}\right]. \qquad \text{(EPA}')$$

(Here, $\mathcal{L}^{(i)}$ *sends a* $j \times j$ *matrix* H *to the* $j \times j$ *matrix having* $i \times i$ *minor equal to* $H^{(i)}$ *and null entries otherwise.)*

Proof. From now on, we will denote the skew-Hermitian matrix in the left entry of the Lie bracket in (EPA′) by $\mathcal{L}^*(H)$.

The dynamical system (EPA′) is well defined in the vector space $\mathfrak{u}(n) \oplus \cdots \oplus \mathfrak{u}(1) \subset \mathfrak{u}(n(n-1)/2)$; the non-trivial part of the lemma consists in proving the $\mathcal{S}$-compatibility of the equations, i.e. consists in showing that the subspace $\mathcal{S}$ is an invariant subspace for the dynamical system.

The compatibility conditions are the identities

$$\begin{aligned}&[H^{(n-i)}, \mathcal{L}^{(n-i-1)}(H^{(n-i)}) + \cdots + \mathcal{L}^{(1)}(H^{(n-i)})]^{(n-i-1)} \\ &\qquad = [H^{(n-i-1)}, \mathcal{L}^{(n-i-2)}(H^{(n-i-1)}) + \cdots + \mathcal{L}^{(1)}(H^{(n-i-1)})]\end{aligned}$$

for $i = 1, \ldots, n-2$. These identities hold because

$$
\begin{aligned}
&[H^{(n-i)}, \mathcal{L}^{(n-i-1)}(H^{(n-i)}) + \cdots + \mathcal{L}^{(1)}(H^{(n-i)})] \\
&\quad = \left[\left(\begin{array}{c|c} H^{(n-i-1)} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline * \cdots * & * \end{array} \right), \left(\begin{array}{c|c} H^{(n-i-1)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 \end{array} \right) + \cdots + \left(\begin{array}{c|c} H^{(1)} & 0 \cdots 0 \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & 0 \end{array} \right) \right] \\
&\quad = \left(\begin{array}{c|c} [H^{(n-i-1)}, \mathcal{L}^{(n-i-1)}(H^{(n-i-1)}) + \cdots + \mathcal{L}^{(1)}(H^{(n-i-1)})] & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline * \quad \cdots \quad * & 0 \end{array} \right) \\
&\quad = \left(\begin{array}{c|c} [H^{(n-i-1)}, \mathcal{L}^{(n-i-2)}(H^{(n-i-1)}) + \cdots + \mathcal{L}^{(1)}(H^{(n-i-1)})] & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline * \quad \cdots \quad * & 0 \end{array} \right).
\end{aligned}
$$

□

Writing a λ-parametric perturbation for equation (EPA′) is very natural. In fact, the differential equation

$$
\begin{aligned}
&\left(\mathcal{H} + \lambda \left(\begin{array}{c|c|c} \mathcal{L}^{(n-1)}(\mathbb{I}_n) & 0 & 0 \\ \hline 0 & \mathcal{L}^{(n-2)}(\mathbb{I}_{n-1}) & 0 \\ \hline 0 & 0 & \ddots \end{array} \right) \right)^{\bullet} \\
&\quad = \left[\mathcal{H} + \lambda \left(\begin{array}{c|c|c} \mathcal{L}^{(n-1)}(\mathbb{I}_n) & 0 & 0 \\ \hline 0 & \mathcal{L}^{(n-2)}(\mathbb{I}_{n-1}) & 0 \\ \hline 0 & 0 & \ddots \end{array} \right), \mathcal{L}^* \right]
\end{aligned}
$$

is equivalent to (EPA′). The characteristic polynomial of the λ-dependent matrix

$$
\left(\begin{array}{c|c|c} H & 0 & 0 \\ \hline 0 & H^{(n-1)} & 0 \\ \hline 0 & 0 & \ddots \end{array} \right) + \lambda \left(\begin{array}{c|c|c} \mathcal{L}^{(n-1)}(\mathbb{I}_n) & 0 & 0 \\ \hline 0 & \mathcal{L}^{(n-2)}(\mathbb{I}_{n-1}) & 0 \\ \hline 0 & 0 & \ddots \end{array} \right)
$$

is the product of the characteristic polynomials of the matrices $H^{(i)} + \lambda\mathcal{L}^{(i-1)}(\mathbb{I}_i)$. The coefficients of the powers of λ and μ in the characteristic polynomial redundantly give a set of functions which are those generated by the functions τ^i_j. We can conclude by stating

Theorem 1.8. *The Hamiltonian system on the manifold $\mathcal{O}_n$ given by the Hamiltonian f can be written as a λ-parametric Lax equation. Such an equation gives rise to the $n(n-1)/2$ integrals of motion that are called the Gelfand–Cetlin system.*

Remark. Equation (EPA′) already implies that the eigenvalues of the minors $H^{(i)}$ are integrals of motion. Having a λ-parametric Lax equation allows one to define a Riemann surface and to use inverse scattering theory to describe the flow of the system.

2. Global action–angle variables

Given a completely integrable system, the question whether there exist global action–angle variables gives rise to some remarkable cohomological considerations pertaining to the theory of obstruction. These cohomology classes have been observed by Nekhoroshev in [15] and fully described by Duistermaat [6] for the commutative case ($d = n$) and by Dazord and

Delzant [4] for the non-commutative case ($d > n$). A thorough treatment can be found in [19]; we recall here the main structures that appear.

A completely integrable system ($K \to M \to P$) gives rise to a covering of P whose monodromy, called *monodromy of the completely integrable system*, is the obstruction to the existence of global multi-valued action variables (closed 1-forms). If global multi-valued action variables exist, then the submersion $M \to P$ is a principal torus bundle; the Chern class of this bundle is (of course) the obstruction to the existence of global topological angle variables. Two other obstructions to global action–angle coordinates are: a family of 1-cohomology classes, related to the exactness of the multi-valued action variables, and, in the commutative case ($d = n$), a 2-cohomology class, related to the existence of global angle variables in which the expression of the symplectic 2-form is as in the theorem on action–angle variables $\sigma = \mathrm{d}\varphi_i \wedge \mathrm{d}a^i$.

From proposition 1.1 and lemma 1.2 one deduces that the submersion $\vec{\mu} : \mathcal{O}'_n \to \mathbb{R}^{n(n-1)/2}$ gives rise to a $T^{n(n-1)/2}$ torus bundle over a space diffeomorphic to $(0, 1)^{n(n-1)/2}$. Given that such a space cannot support monodromy nor non-trivial cohomology classes, it follows that the torus bundle must be trivial, and also that there must exist global circle-valued functions φ^i_p such that the symplectic structure of $\mathcal{O}'_n$ is the closed 2-form $\mathrm{d}\mu^i_p \wedge \mathrm{d}\varphi^i_p$.

Remark. The Gelfand–Cetlin system is given as a family of globally defined angle variables; for this reason we do not need to use the 1-connectedness of the base to state the existence of global action variables. On the other hand, on the vanishing of the second cohomology group of $(0, 1)^{n(n-1)/2}$ we base the claim of the existence of global angle variables.

It is clear that the angle variables must be related to the phases of the entries of a given matrix H in $\mathcal{O}'_n$. The argument of a complex number cannot be defined at zero; this turns out to be the main obstacle in defining globally angle coordinates that, on open subsets, can be easily written.

Lemma 2.1. *Let H be a matrix in $\mathcal{O}'_n$, and let v^i_p be an eigenvector $H^{(i)}$ associated with the eigenvalue μ^i_p, then the last entry of v^i_p is non-zero.*

Proof. Assume that the ith component of v^i_p is zero, then the vector w obtained by v^i_p cancelling the zero is an eigenvalue of $H^{(i-1)}$ with the same eigenvalue of μ^i_p. This can never happen for a matrix in $\mathcal{O}'_n$. □

The above lemma implies that one has a preferred choice for the eigenvector v^i_p, since it can be imposed that the eigenvectors have norm 1, with last component real and positive. This is what we will assume in the rest of this paper.

To give an explicit expression of the Hamiltonian flow of μ^i_p we need to recall a fact proved in [8].

Lemma 2.2. *Let μ be an eigenvalue of a Hermitian matrix $H^{(i)}$, and let u be a normal eigenvector associated with μ. If v is the n-dimensional vector $(u_1, \ldots, u_i, 0, \ldots, 0)^t$, then*

$$\underline{\mu}(H) = v \otimes v^*$$

and hence

$$\mathrm{e}^{t\underline{\mu}} H = \mathrm{e}^{\mathrm{i}tv\otimes v^*} H \, \mathrm{e}^{\mathrm{i}tv\otimes v^*}.$$

In this statement, as in the rest of the paper, we underline a function to indicate its associated Hamiltonian vector field (as Guillemin and Sternberg in [10]); we use the notation of the elementary exponential map to indicate both, the flow of a vector field and the exponential map from the theory of Lie groups.

Let P_i^* denote the matrix $\left(v_1^i, \ldots, v_i^i\right)$ (where the vectors are columns), then

$$\left(\begin{array}{c|c} P_i & 0 \\ \hline 0 & \mathbb{I}_{n-i} \end{array}\right) H \left(\begin{array}{c|c} P_i^* & 0 \\ \hline 0 & \mathbb{I}_{n-i} \end{array}\right) = \left(\begin{array}{ccc|ccc} \mu_1^i & & 0 & k_{1,i+1} & \cdots & k_{1,n} \\ & \ddots & & \vdots & & \vdots \\ 0 & & \mu_i^i & k_{i,i+1} & \cdots & k_{i,n} \\ \hline k_{i+1,1} & \cdots & k_{1+1,i} & k_{i+1,i+1} & \cdots & k_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ k_{n,1} & \cdots & k_{n,i} & k_{n,i+1} & \cdots & k_{n,n} \end{array}\right) = K.$$

The Hamiltonian flow of the function μ_p^i is

$$\mathrm{e}^{t\underline{\mu_p^i}} H = \left(\begin{array}{c|c} P_i^* & 0 \\ \hline 0 & \mathbb{I}_{n-i} \end{array}\right) \begin{pmatrix} \mathbb{I}_{p-1} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}t} & 0 \\ 0 & 0 & \mathbb{I}_{n-p} \end{pmatrix} K \begin{pmatrix} \mathbb{I}_{p-1} & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t} & 0 \\ 0 & 0 & \mathbb{I}_{n-p} \end{pmatrix} \left(\begin{array}{c|c} P_i & 0 \\ \hline 0 & \mathbb{I}_{n-i} \end{array}\right).$$

Hence, one can read the phases conjugate to the action functions μ_p^i from the argument of any non-zero entry in the pth line of the matrix K. This definition is not global since, with the choice of an entry in the pth row, we have implicitly assumed that this entry is non-zero. The fact that H belongs to $\mathcal{O}_n'$ imposes another condition on the matrix K, namely that the complex numbers $k_{p,i+1}$ are never zero. In fact, if $k_{p,i+1}$ was zero for some p, then the eigenvalue μ_p^i would also be an eigenvalue for the matrix $H^{(i+1)}$, and this is forbidden.

We can hence focus our attention to the phases of the complex numbers $k_{p,i+1}$ and define

$$\varphi_p^i = \arg(k_{p,i+1}) = \arg\left(\left(v_p^i\right)^* \begin{pmatrix} h_{1,i+1} \\ \vdots \\ h_{i,i+1} \end{pmatrix}\right) \tag{2}$$

to be global angle coordinates for the Gelfand–Cetlin system. To conclude the section, we need to check the commutation relations between the eigenvalues μ_p^i and the circle-valued functions φ_p^i and among the angle coordinates themselves.

Lemma 2.3. *Let p, q be integers between 1 and i, then*

$$\left\{\mu_p^i, \varphi_q^i\right\} = \delta_{pq}.$$

The proof of this lemma follows straightforwardly from the definition of angle variables. The other commutation relations are a little more delicate.

Lemma 2.4. *Let $i \neq j$, q an integer between 1 and i and p an integer between 1 and j, then*

$$\left\{\mu_p^i, \varphi_q^j\right\} = 0.$$

Proof. There are two cases that require different treatments. In the first case, $i > j$, the computation is easy. In fact, the matrix $H^{(j)}$ does not change along the flow of μ_p^i; so, also the eigenvector v_q^j must be constant along the μ_p^i-flow. The column $(h_{1,j+1} \cdots h_{j,j+1})^t$ is also not changed by the flow of μ_p^i. It hence follows that the function φ_q^j commutes with μ_p^i.

The second case, $i < j$, is a little more difficult. In this case, the eigenvalue μ_q^j is of course constant (it Poisson-commutes with the eigenvalue μ_p^i) but the eigenvector v_q^j is not constant. A calculation shows that the evolution of v_q^j along the flow of μ_p^i is

$$v_q^j\left(\mathrm{e}^{t\underline{\mu_p^i}}H\right) = \left(\begin{array}{c|c} P_i^* \begin{pmatrix} \mathbb{I}_{p-1} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}t} & 0 \\ 0 & 0 & \mathbb{I}_{i-p} \end{pmatrix} P_i & 0 \\ \hline 0 & \mathbb{I}_{j-i} \end{array}\right) v_q^j.$$

We have the expression of the eigenvector v_q^j along the flow of the action function μ_j^i. Let us be more explicit on the form of the column $(h_{1,j+1} \cdots h_{j,j+1})^t$ along the flow of the Hamiltonian μ_p^i. We hence need to compute the $(j+1)$th column of the matrix

$$\left(\begin{array}{c|c} P_i^* & 0 \\ \hline 0 & \mathbb{I}_{n-i} \end{array}\right) \left(\begin{array}{ccccc|ccc} \mu_1^i & & & & & k_{1,i+1} & \cdots & k_{1,n} \\ & \ddots & & 0 & & \vdots & & \vdots \\ & & \mu_p^i & & & \mathrm{e}^{\mathrm{i}t} k_{p,i+1} & \cdots & \mathrm{e}^{\mathrm{i}t} k_{p,n} \\ & 0 & & \ddots & & \vdots & & \vdots \\ & & & & \mu_i^i & k_{i,i+1} & \cdots & ki,n \\ \hline k_{i+1,1} & \cdots & \mathrm{e}^{-\mathrm{i}t} k_{i+1,p} & \cdots & k_{i+1,i} & k_{i+1,i+1} & \cdots & k_{i+1,n} \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ k_{n,1} & \cdots & \mathrm{e}^{-\mathrm{i}t} k_{n,p} & \cdots & k_{n,i} & k_{n,i+1} & \cdots & k_{n,n} \end{array}\right) \left(\begin{array}{c|c} P_i & 0 \\ \hline 0 & \mathbb{I}_{n-i} \end{array}\right)$$

which is the column

$$\left(\begin{array}{c|c} P_i^* & 0 \\ \hline 0 & \mathbb{I}_{j-i} \end{array}\right) \begin{pmatrix} k_{1,j+1} \\ \vdots \\ \mathrm{e}^{\mathrm{i}t} k_{p,j+1} \\ \vdots \\ k_{j,j+1} \end{pmatrix}.$$

One can finally compute the derivative of φ_q^j along the vector field $\underline{\mu_p^i}$,

$$\arg\left(\left(v_q^j\right)^* \left(\begin{array}{c|c} P_i^* \begin{pmatrix} \mathbb{I}_{p-1} & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t} & 0 \\ 0 & 0 & \mathbb{I}_{i-p} \end{pmatrix} P_i & 0 \\ \hline 0 & \mathbb{I}_{j-i} \end{array}\right) \left(\begin{array}{c|c} P_i^* & 0 \\ \hline 0 & \mathbb{I}_{j-i} \end{array}\right) \begin{pmatrix} k_{1,j+1} \\ \vdots \\ \mathrm{e}^{\mathrm{i}t} k_{p,j+1} \\ \vdots \\ k_{j,j+1} \end{pmatrix} \right)$$

$$= \arg\left(\left(v_q^j\right)^* \left(\begin{array}{c|c} P_i^* \begin{pmatrix} \mathbb{I}_{p-1} & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t} & 0 \\ 0 & 0 & \mathbb{I}_{i-p} \end{pmatrix} & 0 \\ \hline 0 & \mathbb{I}_{j-i} \end{array}\right) \begin{pmatrix} k_{1,j+1} \\ \vdots \\ \mathrm{e}^{\mathrm{i}t} k_{p,j+1} \\ \vdots \\ k_{j,j+1} \end{pmatrix} \right)$$

and obtain their constancy. □

Remark. The commutation relations so far proved imply that the functions φ_p^i are linearly independent at every point of $\mathcal{O}_n'$.

The last fact to check is that the functions φ_p^i Poisson-commute. Following the ideas in [8], we need to prove that the restriction of the symplectic form of $\mathcal{O}_n'$ to the level set $\mathcal{L} = \left\{\varphi_p^i = 0, i = 1, \ldots, n-1, p = 1, \ldots, i\right\}$ is zero. To prove this we need a lemma.

Lemma 2.5. *The manifold $\mathcal{L}$ is the set of matrices of $\mathcal{O}_n'$ with real entries.*

Proof. Let H be a real Hermitian matrix, then H is a symmetric real matrix and its eigenvectors v_p^i have real entries. It follows that the angle functions $\varphi_p^i(H)$ are all equal to 1.

Conversely, let H be a matrix in $\mathcal{O}_n'$ such that $\varphi_p^i(H) = 1$ for all i and p. The angle $\varphi_1^1(H)$ is 1 if and only if the entry $h_{1,2}$ of H is real; it follows that the eigenvectors v_1^2 and v_2^2 of $H^{(2)}$ have real entries. The angles $\varphi_1^2(H)$ and $\varphi_2^2(H)$ are 1 if and only if the vector $(h_{1,3}, h_{2,3})^t$ is a real-coefficient combination of the vectors v_1^2, v_2^2, and hence has real entries. Iteration of this argument proves that H must be a matrix with real coefficients. □

Proposition 2.6. *The sub-manifold $\mathcal{L}$ is a Lagrangian manifold of $\mathcal{O}_n'$.*

Proof. The vectors tangent to $\mathcal{L}$ at one of its points H are only and all the vectors $\mathrm{ad}_X H = [X, H]$ with X a real anti-symmetric matrix. Using the expression for the symplectic form given in section 1.2, one obtains that $\sigma_H(\mathrm{ad}_X, H, \mathrm{ad}_Y H) = \mathrm{i}\,\mathrm{Tr}(H[X, Y]) = \mathrm{Tr}(YHX) - \mathrm{Tr}(XHY)$. But $(YHX)^t = X^t H^t Y^t = XHY$, and this proves that the manifold $\mathcal{L}$ is Lagrangian. □

Remark. Hausmann and Knutson [12] wrote an isomorphism between the symplectic reduction of a regular coadjoint orbit by the Horn–Schur T^{n-1}-action and a subspace of Flaschka and Millson's polygon space. This isomorphism identifies the action functions of the Gelfand–Cetlin system with the action functions of the bending flow (the length of some of the diagonals of the polygon). The same isomorphism identifies the angle functions we just described with the four-point formula in [8].

3. The superintegrable Gelfand–Cetlin system: some cohomology

The definition of a superintegrable Gelfand–Cetlin system appears implicitly in the work of Guillemin and Sternberg. In fact, in [11] the authors show that the Gelfand–Cetlin system can be obtained using a recursion; the first step of this recursion amounts to observing the existence of a non-commutative completely integrable system in $\mathcal{O}_n$. The fibres of this integrable system are tori of dimension $n-1$, while the base of the bundle is the direct product of $n-1$ action variables and a regular $U(n-1)$-coadjoint orbit $\mathcal{O}_{n-1}$. The recursion proceeds by obtaining $n-2$ commuting functions defined on $\mathcal{O}_{n-1}$.

The completely integrable system which we are about to describe is the first step in Guillemin and Sternberg's recursion, and is a non-commutative completely integrable system which presents a non-vanishing Chern class.

The symplectic manifold $\mathcal{O}_n$, being a $U(n)$-coadjoint orbit, is a homogeneous $U(n)$-space. The group $U(n-1)$ can be embedded in $U(n)$ as the set of unitary transformations that fix the last basis vector (once a basis is fixed); hence, the space $\mathcal{O}_n$ is endowed with a Hamiltonian $U(n-1)$-action. The momentum map associated with such action is the projection $\mathcal{O}_n \subset \mathrm{i}\mathfrak{u}(n)$ in $\mathrm{i}\mathfrak{u}(n-1)$ obtained by cancelling the last row and column of a matrix H in $\mathcal{O}_n$

$$\mathcal{O}_n \xrightarrow{\pi} \mathrm{i}\mathfrak{u}(n-1) \qquad H \mapsto H^{(n-1)}.$$

This map is a Poisson morphism.

The restriction of π to the set of its regular points is a T^{n-1} fibration over an open set of the Poisson manifold $\mathrm{i}\mathfrak{u}(n-1)$ and defines a completely integrable system. We begin the investigation of this completely integrable system by first describing the Poisson manifold which is the base of the submersion.

Proposition 3.1. *The image of the map π is the set $\{K \in \mathfrak{u}^*(n-1) | \mu_i^{n-1}(K) \in [\mu_i, \mu_{i+1}]\}$. The critical values of π are given by the equations $\mu_i^{n-1} = \mu_i$ and $\mu_i^{n-1} = \mu_{i+1}$.*

Proof. Given a Hamiltonian group action, the rank of the momentum map at a point is the dimension of the orbit through that point. Let H be a Hermitian matrix, and let $P \in U(n-1)$ be such that $PH^{(n-1)}P^*$ is diagonal. The matrices of $U(n-1)$ that stabilize $H^{(n-1)}$ are

$$\left(\begin{array}{c|c} PT^{n-1}P^* & 0 \\ \hline 0 & 1 \end{array}\right).$$

Hence, if

$$H = \left(\begin{array}{c|c} H^{(n-1)} & h \\ \hline h^* & a \end{array}\right)$$

the pre-image of $H^{(n-1)}$ is an $(n-1)$-torus if and only if Ph is a vector with non-vanishing entries. But Ph has non-vanishing entries if and only if some of the eigenvalues of $H^{(n-1)}$ are eigenvalues of H. □

Let $\mathcal{O}_n''$ be the set of regular points for π; it can be shown that $\mathcal{O}_n''$ is connected and it is obvious that the image $P = \pi(\mathcal{O}_n'')$ is an open set of $\mathrm{i}\mathfrak{u}(n-1)$. The open set P, as any open subset of $\mathrm{i}\mathfrak{u}_{\mathrm{reg}}(n)$, inherits a regular Poisson structure of rank $(n-1)(n-2)$, and is a trivial bundle over an action space which is precisely the intersection of P with a Weyl chamber of $\mathrm{i}\mathfrak{u}(n-1)$. This space is the set $A = (\mu_1, \mu_2) \times \cdots \times (\mu_{n-1}, \mu_n)$. The diagram

$$(T^{n-1} \to)\mathcal{O}_n'' \to P(\to A)$$

defines a non-commutatively completely integrable system. Also in this case the existence of globally defined action functions implies the vanishing of the monodromy of this system, i.e. the principality of the torus bundle. On the other hand, the Poisson manifold P has non-trivial second cohomology group, and can hence support obstructions to global angle variables–a Chern class.

We will compute the Chern class as it is defined in [5]. The Chern class is the obstruction to the existence of a section of a principal torus bundle. One can try to build such a section by CW-decomposing the base manifold, defining a section above the 0-cells, extending it consistently over the 1-cells, and so on with higher dimensional cells. When dealing with a T^{n-1}-bundle, the first (and only) obstruction appears when trying to extend the section to the 2-cells, and is a map from such 2-cells to the fundamental group of the torus $\mathbb{Z}^{n-1}$.

In our specific case, the base manifold retracts on the manifold $\mathcal{O}_{n-1}$, and there is a natural CW-decomposition of the manifold $\mathcal{O}_{n-1}$, known as Bruhat decomposition. This decomposition has only even cells: the 0-cell is a chosen point K_0 of $\mathcal{O}_{n-1}$, the 2-cells are obtained by conjugating the point K_0 by matrices of the form

$$\left(\begin{array}{c|cc|c} \mathbb{I}_{i-1} & 0 & 0 & 0 \\ \hline 0 & s(z) & c(z) & 0 \\ 0 & -c(z) & \bar{s}(z) & 0 \\ \hline 0 & 0 & 0 & \mathbb{I}_{n-i-2} \end{array}\right)$$

with $s(z) = z/\sqrt{1+|z|^2}$ and $c(z) = 1/\sqrt{1+|z|^2}$. More generally, the $2p$-cells are obtained by conjugating K_0 by appropriate products of matrices of the above form. This choice of parameters for the Bruhat cells can be found in [13].

Choosing

$$H_0 = \left(\begin{array}{ccc|c} \mu_1^{n-1} & & 0 & h_1 \\ & \ddots & & \vdots \\ 0 & & \mu_{n-1}^{n-1} & h_{n-1} \\ \hline h_1^* & \cdots & h_{n-1}^* & h_{n,n} \end{array}\right)$$

as the section above the base point $K_0 = H_0^{(n-1)}$ in $\mathrm{iu}(n-1)$, one has that a basis of the fundamental group of the fibre above K_0 is generated by the paths

$$\gamma_i(t) = \begin{pmatrix} \mathbb{I}_{i-1} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}t} & 0 \\ 0 & 0 & \mathbb{I}_{n-i} \end{pmatrix} H_0 \begin{pmatrix} \mathbb{I}_{i-1} & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t} & 0 \\ 0 & 0 & \mathbb{I}_{n-i} \end{pmatrix}$$

and the 2-cell σ_1 is parametrized by

$$z \mapsto \left(\begin{array}{cc|ccc} |s(z)|^2\mu_1^{n-1} + c^2(z)\mu_2^{n-1} & c(z)s(z)\left(\mu_2^{n-1} - \mu_1^{n-1}\right) & 0 & \cdots & 0 \\ c(z)\bar{s}(z)\left(\mu_2^{n-1} - \mu_1^{n-1}\right) & |s(z)|^2\mu_2^{n-1} + c^2(z)\mu_1^{n-1} & 0 & \cdots & 0 \\ \hline 0 & 0 & \mu_3^{n-1} & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & \mu_{n-1}^{n-1} \end{array}\right)$$

with z a complex number.

Proposition 3.2. *The Chern class of the Gelfand–Cetlin non-commutative system is the cocycle that associates to the Bruhat 2-cells σ_i, $i = 1, \ldots, n-2$, the cycles $\gamma_i - \gamma_{i+1}$.*

Proof. All we need to do is to compute a section above the 2-cell σ_1; in fact, the expression of a section above the other 2-cells is very similar. By conjugating the matrix H_0 by the matrix

$$\left(\begin{array}{cc|c} s(z) & c(z) & 0 \\ -c(z) & \bar{s}(z) & 0 \\ \hline 0 & 0 & \mathbb{I}_{n-2} \end{array}\right)$$

one writes a natural section above the cell σ_1,

$$\left(\begin{array}{ccc|c} |s(z)|^2\mu_1^{n-1} + c(z)^2\mu_2^{n-1} & s(z)c(z)\left(\mu_2^{n-1} - \mu_1^{n-1}\right) & & s(z)h_1 + c(z)h_2 \\ c(z)\bar{s}(z)\left(\mu_2^{n-1} - \mu_1^{n-1}\right) & |s(z)|^2\mu_2^{n-1} + c(z)^2\mu_1^{n-1} & & \bar{s}(z)h_2 - c(z)h_1 \\ & & \ddots & \vdots \\ \hline \bar{s}(z)\bar{h}_1 + c(z)\bar{h}_2 & s(z)\bar{h}_2 - c(z)\bar{h}_1 & \cdots & h_{n,n} \end{array}\right).$$

Letting $z = \mathrm{e}^{\mathrm{i}\varphi}\rho$ and ρ tend to infinity, the given section draws the cycle $\gamma_1 - \gamma_2$. □

The non-vanishing of the Chern class can also be proved using a different argument. In [4], Dazord and Delzant have proved that the Chern class of a completely integrable system maps in the characteristic class form of the base Poisson manifold. But, the characteristic class form of any Ad_G^*-invariant open set of $\mathfrak{g}_{\mathrm{reg}}^*$ is non-zero if G is semisimple, hence the Chern class cannot vanish. For the definition of characteristic class form and for a sketch of the proof that the characteristic class form of the dual of a semisimple Lie algebra does not vanish, we refer to the book [19].

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